NONLINEAR HEAT CONDUCTION WITH NONSTATIONARY BOUNDARY CONDITIONS

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We study the one-dimensional nonstationary temperature field in a solid when the thermal conductivity and heat capacity depend linearly on the temperature.

In making thermal calculations for conditions of intense thermal activity it becomes necessary to consider nonlinear problems of nonstationary heat conduction because of the need to account for the dependence of heat transfer coefficients on the temperature. The solution of such problems, even in the simplest cases, for example, for stationary boundary conditions of the first kind, is very difficult. However, it is often necessary in practice to have an analytical description of the temperature distribution for the case of a time-varying heat transfer coefficient and also a description of the temperature of the surrounding medium. This circumstance complicates the problem substantially since, in addition to the nonlinearity of the boundary conditions arising from the dependence of the coefficient of thermal conductivity on the temperature of the surface of the body, the nonstationarity of these boundary conditions must also be taken into account.

A problem of this type may be formulated as follows:

$$\frac{\partial}{\partial x}\left[\lambda(t)\frac{\partial t(x,\ \tau)}{\partial x}\right]=\rho C(t)\frac{\partial t(x,\ \tau)}{\partial \tau}, \tag{1}$$

$$\left.\frac{\partial t(x,\ \tau)}{\partial x}\right|_{x=0}=0, \tag{2}$$

$$\lambda(t)\left.\frac{\partial t(x,\ \tau)}{\partial x}\right|_{x=\delta}=\alpha(\tau)\left[t_c(\tau)-t(\delta,\ \tau)\right], \tag{3}$$

$$t(x,\ 0)=0.$$

A strictly analytical solution of the problem defined by Eqs. (1)-(3) is not possible. Only approximate methods of various kinds are available for problems of this type (see [1-5] and other references).

A solution to problem (1) was given in [1] for the case of stationary boundary conditions of the first and third kinds and with the assumption of linear dependence on the temperature of the heat capacity and the thermal conductivity. The integral relations due to L. S. Leibenzon were used in obtaining the solution. In [2] a solution was obtained for the case of nonstationary boundary conditions of the first kind. By dividing the time duration of the process and the plate thickness into a number of computational intervals the author reduced the problem to a linear one for a multilayered system. In [3] a numerical method was given.

The most complete treatment of the problem (1)-(3) was given in [4]. The solution was obtained as a series in powers of a small parameter ε, assuming the heat capacity and thermal conductivity to depend linearly on the temperature. However, in handling the conditions (2) the author placed a restriction on the manner of variation of the heat transfer coefficient $\alpha(\tau)$ and on the temperature $t_c(\tau)$ of the medium, which he took to be constant in time.

In this paper we consider an approximate method for solving problem (1)-(3) when the heat transfer coefficient and the temperature of the surrounding medium vary arbitrarily with the time. We obtain our solution with the aid of the special bilinear series given in [5], guaranteeing thereby rapid convergence.

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With an increase in temperature the heat capacity C of metals increases while the thermal conductivity coefficient λ may increase or decrease. In the last case, the effect of variability of the thermal properties is partially or even completely compensated for [6], in the first case, however, it may turn out to be significant. In accordance with this, then, we consider the first case.

Based on the data given in [7], we may assume that the quantities $\lambda(t)$ and C(t) vary linearly with the temperature:

$$\lambda(t) = \lambda_0 - \lambda_1 \frac{t(x,\tau)}{t_c(\tau)}, \qquad C(t) = C_0 + C_1 \frac{t(x,\tau)}{t_c(\tau)}. \tag{4}$$

We introduce the dimensionless parameters [6]

$$\xi = \frac{x}{\delta}; \quad \mathrm{Bi}(\tau) = \frac{\alpha(\tau)\,\delta}{\lambda_0 - \lambda_1}; \quad \mathrm{Fo} = \frac{\lambda_0-\lambda_1}{C_0+C_1}\cdot\frac{\tau}{\rho\delta^2}; \quad \Theta = \frac{t(x,\tau)}{t_c(\tau)} - 1. \tag{5}$$

Substituting the relations (4) and (5) into the set of Eqs. (1)-(3), we obtain, after some simplifications,

$$\frac{\partial^2 f_1(\Theta)}{\partial\xi^2} = \frac{\partial f_2(\Theta)}{\partial \mathrm{Fo}} + \bar{t}_c\,[f_3(\Theta)+1],$$

$$\left.\frac{\partial\Theta}{\partial\xi}\right|_{\xi=0} = 0,$$

$$\left.\frac{\partial\Theta}{\partial\xi}\right|_{\xi=1} = \mathrm{Bi}(\mathrm{Fo})\, f_4(\Theta)\,|_{\xi=1},$$

where

$$f_1(\Theta) = \Theta\left(1 - \frac{1}{2}\cdot\frac{\lambda_1}{\lambda_0-\lambda_1}\Theta\right); \quad f_2(\Theta) = \Theta\left(1 + \frac{1}{2}\cdot\frac{C_1}{C_0+C_1}\Theta\right);$$

$$f_3(\Theta) = \Theta\left[1 + \frac{C_1}{C_0+C_1}(1+\Theta)\right]; \quad f_4(\Theta) = \frac{\Theta}{1 - \frac{\lambda_1}{\lambda_0-\lambda_1}\Theta};$$

$$\bar{t}_c = \frac{1}{t_c(\tau)}\cdot\frac{dt_c(\tau)}{d\tau}.$$

The graphs in Fig. 1 show the variation of $f_1(\Theta)$, $f_2(\Theta)$, $f_3(\Theta)$ and $f_4(\Theta)$ with Θ for the whole range of values $<-1, 0>$. These graphs were drawn assuming that $\lambda_1 = 0.5\lambda_0$, $C_1 = C_0$. Such values correspond to a decrease of $\lambda(t)$ by 50% and an increase of C(t) by 100% in comparison with their initial values, for the attainment of the temperature equal to the temperature of the surrounding medium, and this characterizes a wide range of variation of the thermal properties of materials.

As is evident from the graphs, the curves $f(\Theta)$ may be approximated fairly precisely by straight lines:

$$f_j(\Theta) = K_j\Theta.$$

The values of K_j may be determined from the relations

$$\int_{-1}^{0} [f_j(\Theta) - K_j\Theta]\,d\Theta = 0.$$

Upon making such an approximation we obtain the linear problem

$$K_1\frac{\partial^2\Theta}{\partial\xi^2} = K_2\frac{\partial\Theta}{\partial \mathrm{Fo}} + (K_3\Theta + 1), \tag{6}$$

$$\left.\frac{\partial\Theta}{\partial\xi}\right|_{\xi=0} = 0,$$

$$\left.\frac{\partial\Theta}{\partial\xi}\right|_{\xi=1} = -\,\mathrm{Bi}(\mathrm{Fo})\,K_4\Theta\,|_{\xi=1},$$

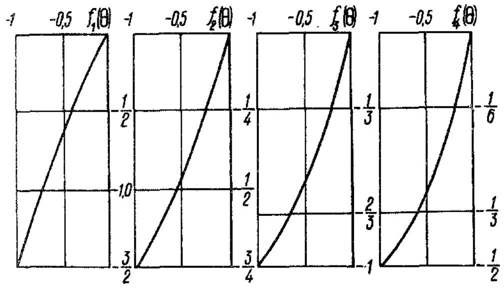

Fig. 1. Variation of f_1, f_2, f_3, and f_4 with Θ.

$$\Theta|_{\mathrm{Fo}=0} = -1, \tag{7}$$

where

$$K_1 = 1 + \frac{1}{3}\frac{\lambda_1}{\lambda_0 - \lambda_1}; \quad K_2 = 1 - \frac{1}{3}\frac{C_1}{C_0 + C_1};$$
$$K_3 = 1 + \frac{1}{6}\frac{C_1}{C_0 + C_1}; \tag{8}$$
$$K_4 = 2\frac{\lambda_0 - \lambda_1}{\lambda_1}\left(1 - \frac{\lambda_0 - \lambda_1}{\lambda_1}\ln\frac{\lambda_0}{\lambda_0 - \lambda_1}\right).$$

The solution of the problem (6)-(7) was given in [8]; we give here only the final result:

$$\Theta = -\sum_{i=1}^{n} \gamma_i(\xi, \mathrm{Fo})\,\varphi_i(\mathrm{Fo}). \tag{9}$$

In determining $\gamma_i(\xi, \mathrm{Fo})$ it is necessary to take into account that in this case the Green's function is given by

$$G(\xi, \eta, \mathrm{Fo}) = \begin{cases} \dfrac{1}{K_4\mathrm{Bi}(\mathrm{Fo})} + 1 - \eta, & \xi \leqslant \eta; \\ \dfrac{1}{K_4\mathrm{Bi}(\mathrm{Fo})} + 1 - \xi, & \xi \geqslant \eta. \end{cases}$$

The functions $\varphi_i(\mathrm{Fo})$ in Eq. (9) are determined by solving the system of equations

$$\sum_{k,i=1}^{n} A_{k,i}(\mathrm{Fo})\dot{\varphi}_i(\mathrm{Fo}) + [B_{k,i}(\mathrm{Fo}) + \dot{\bar{t}}_c A_{k,i}(\mathrm{Fo})]\,\varphi_i(\mathrm{Fo}) + \varphi_k(\mathrm{Fo}) = \dot{\bar{t}}_c D_k(\mathrm{Fo}) \tag{10}$$

with the initial conditions

$$\varphi_k(0) = \frac{\int_0^1 P_k^*(\eta)\,d\eta}{\int_0^1 \gamma_k(\eta, 0)\,P_k^*(\eta)\,d\eta}. \tag{11}$$

In Eqs. (10)

$$A_{k,i}(\mathrm{Fo}) = \frac{K_2}{K_1}\int_0^1 \gamma_k(\eta, \mathrm{Fo})\,\gamma_i(\eta, \mathrm{Fo})\,d\eta,$$
$$B_{k,i}(\mathrm{Fo}) = \frac{K_2}{K_1}\int_0^1 \gamma_k(\eta, \mathrm{Fo})\,\gamma_i(\eta, \mathrm{Fo})\,d\eta,$$
$$D_k(\mathrm{Fo}) = \int_0^1 \gamma_k(\eta, \mathrm{Fo})\,d\eta.$$

By using in the solution a special uniformly convergent series, we can, in practice, limit ourselves in Eq. (9) to two or even only one term of the sum.

In the second case, after some simplifications, the solution of the problem has the form

$$\Theta = -\frac{\dfrac{1}{K_4\mathrm{Bi}(\mathrm{Fo})} + \dfrac{1}{2}(1 - \xi^2)}{\dfrac{1}{K_4\mathrm{Bi}(\mathrm{Fo})} + \dfrac{1}{3}}\left\{\frac{1}{K_2}\int_0^{\mathrm{Fo}} \dot{\bar{t}}_c \exp\left[\int_0^{\mathrm{Fo}}\left(\frac{K_1}{K_2 A_{1,1}}\right.\right.\right.$$

$$+\frac{K_3}{K_2}\dot{t}_c\Big)\,d\mathrm{Fo}\Big]\,d\mathrm{Fo}+1\Big\}\exp\Big[-\int\limits_0^{\mathrm{Fo}}\Big(\frac{K_1}{K_2A_{1,1}}+\frac{K_3}{K_2}\dot{t}_c\Big)\,d\mathrm{Fo}\Big], \tag{12}$$

where

$$\frac{1}{A_{1,1}}=\frac{3K_4\mathrm{Bi}\,(\mathrm{Fo})}{3+K_4\mathrm{Bi}\,(\mathrm{Fo})}\,;\ \mathrm{Bi}\,(\mathrm{Fo})=\frac{\alpha\,(\mathrm{Fo})\,\delta}{\lambda_0-\lambda_1}. \tag{13}$$

In accord with Eqs. (5) and (12) we finally obtain

$$t\,(x,\ \tau)=t_c\,(\tau)\,(\Theta+1). \tag{14}$$

Still another method can be used to linearize the problem (1)-(3).

The following method of linearization was given in [9] for the condition C = const.

If in the system (1), (2) we put

$$\rho C=\mathrm{const},\ \lambda\,(t)\frac{\partial t}{\partial\lambda}=m,\quad \frac{\partial t}{\partial\lambda}=n \tag{15}$$

and introduce the new variable

$$z=\lambda-\frac{t_c(\tau)}{n}-\lambda_0, \tag{16}$$

we obtain a linear problem with homogeneous boundary conditions

$$\frac{\partial^2 z}{\partial x^2}=\frac{\rho C}{m}\Big(\frac{\partial z}{\partial\tau}\,n-\dot{t}_c\Big), \tag{17}$$

$$\frac{\partial z}{\partial x}\Big|_{x=0}=0,$$

$$\frac{\partial z}{\partial x}\Big|_{x=\delta}=-\frac{\alpha\,(\tau)\,\delta n}{m}\,z\,(\delta,\ \tau), \tag{18}$$

where

$$\dot{t}_c=\frac{dt_c(\tau)}{d\tau}.$$

We note that the first condition in Eqs. (16) corresponds to an exponential form for the curve λ(t):

$$m=\frac{t_1-t_0}{\ln\lambda_0/\lambda_1}, \tag{19}$$

and the second to a linear approximation for λ(t):

$$n=\frac{t_1-t_0}{\lambda_1-\lambda_0}. \tag{20}$$

The solution of the problem (17), (18) is well known [8]:

$$z\,(x,\ \tau)=-\sum_{i=1}^{n}\gamma_i\,(x,\ \tau)\,\varphi_i\,(\tau). \tag{21}$$

If in Eq. (21) we limit ourselves to only the first term in the sum, we obtain finally

$$t\,(x,\ \tau)=t_c\,(\tau)-\frac{1+\frac{1}{2}\,\mathrm{Bi}\,(\tau)\Big(1-\frac{x^2}{\delta^2}\Big)}{1+\frac{1}{3}\,\mathrm{Bi}\,(\tau)}\Big[\int\limits_0^{\tau}\dot{t}_c\exp\Big(\int\limits_0^{\tau}\frac{d\tau}{A_{1,1}}\Big)\,d\tau+t_c(0)\Big]\exp\Big(-\int\limits_0^{\tau}\frac{d\tau}{A_{1,1}}\Big). \tag{22}$$

The value of $A_{1,1}$ and the Biot number may be determined from the expressions

$$A_{1,1} = \frac{C\rho n}{m}\delta^2\left[\frac{1}{\mathrm{Bi}(\tau)} + \frac{1}{3}\right],$$

$$\mathrm{Bi}(\tau) = \frac{\alpha(\tau)\,\delta n}{m},$$

where m and n are given in Eqs. (19) and (20).

NOTATION

$t(x, \tau)$	is the current temperature;
τ	is the time;
x	is the coordinate;
δ	is the thickness;
ρ	is the density;
λ	is the thermal conductivity;
C	is the heat capacity;
$\alpha(\tau)$, $t_c(\tau)$	are the heat transfer coefficient and temperature of the ambient medium.

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